Physics 3 Exam Test

June 1, 2012

Useful constants: $h = 6.63\,10^{-34} Js, e = 1.6\,10^{-19} C, m_e = 9.1\,10^{-31} kg,$
$L_A = 6.022\,10^{23} 1/mol, c = 2,99792458 10^8 m/s$

1. When a metal is illuminated by light of a suitable wavelength λ it emits electrons. The emission can be prevented by applying an external voltage U to the metal. Determine the value of the Planck constant from the following data: For $\lambda = 279\,nm$ $U = 0.66\,V$ and for $\lambda = 245\,nm$ $U = 1.26\,V$ (**2 points**)

   Solution:

$$h\,\nu = W + \frac{1}{2}mv^2 \text{ and } \lambda = \frac{c}{\nu} \;\Rightarrow\; \frac{h\,c}{\lambda} = W + \frac{1}{2}mv^2$$

When we apply a voltage of U which prohibits the emission then the velocities will be 0 and

$$\frac{h\,c}{\lambda} = W - e\,U$$

From the data:

$$\frac{h\,c}{\lambda_1} = W - e\,U_1, \frac{h\,c}{\lambda_2} = W - e\,U_2$$

Eliminating W:

$$h\,c\,(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}) = e\,(U_2 - U_1)$$

$$h = c\,(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}) = \frac{e\,(U_2 - U_1)}{c\,(\frac{1}{\lambda_1} - \frac{1}{\lambda_2})}$$

$$h = \frac{e\,(U_2 - U_1)\,\lambda_1\,\lambda_2}{c\,(\lambda_2 - \lambda_1)} = 6.447 \cdot 10^{-034} Js$$

2. Calculate the commutator $\left[\hat{L}_x,\, \hat{L}_y\right]$ assuming (correctly) that the classical formula for the angular momentum can be used with operators in quantum mechanics! (**2 points**)

   Solution:

   The classical formula written with operators instead of functions

$$\hat{L}_x = \hat{y} \cdot \hat{p}_z - \hat{z} \cdot \hat{p}_y$$
$$\hat{L}_y = \hat{z} \cdot \hat{p}_x - \hat{x} \cdot \hat{p}_z$$

Solution I.

Let's use the formula:

$$[\hat{x}_j,\, \hat{p}_k] = i\,\hbar\,\delta_{jk}$$

$$\left[\hat{L}_x,\, \hat{L}_y\right] = \hat{L}_x \cdot \hat{L}_y - \hat{L}_y \cdot \hat{L}_x$$

$$= (\hat{y}\cdot\hat{p}_z - \hat{z}\cdot\hat{p}_y)\cdot(\hat{z}\cdot\hat{p}_x - \hat{x}\cdot\hat{p}_z) - (\hat{z}\cdot\hat{p}_x - \hat{x}\cdot\hat{p}_z)\cdot(\hat{y}\cdot\hat{p}_z - \hat{z}\cdot\hat{p}_y)$$

$$= \hat{y}\cdot\hat{p}_z\cdot\hat{z}\cdot\hat{p}_x - \hat{y}\cdot\hat{p}_z\cdot\hat{x}\cdot\hat{p}_z - \hat{z}\cdot\hat{p}_y\cdot\hat{z}\cdot\hat{p}_x + \hat{z}\cdot\hat{p}_y\cdot\hat{x}\cdot\hat{p}_z$$

$$-\hat{z}\cdot\hat{p}_x\cdot\hat{y}\cdot\hat{p}_z + \hat{z}\cdot\hat{p}_x\cdot\hat{z}\cdot\hat{p}_y + \hat{x}\cdot\hat{p}_z\cdot\hat{y}\cdot\hat{p}_z - \hat{x}\cdot\hat{p}_z\cdot\hat{z}\cdot\hat{p}_y$$

$$= \hat{y}\cdot\hat{p}_x\cdot\underbrace{(\hat{p}_z\cdot\hat{z} - \hat{z}\cdot\hat{p}_z)}_{-i\,\hbar} + \underbrace{(-\hat{y}\cdot\hat{p}_z\cdot\hat{x}\cdot\hat{p}_z + \hat{x}\cdot\hat{p}_z\cdot\hat{y}\cdot\hat{p}_z)}_{=0} +$$

$$\underbrace{\hat{z}\cdot\hat{p}_y\cdot\hat{z}\cdot\hat{p}_x - \hat{z}\cdot\hat{p}_x\cdot\hat{z}\cdot\hat{p}_y)}_{=0} + \hat{p}_y\cdot\hat{x}\cdot\underbrace{(\hat{z}\cdot\hat{p}_z - \hat{p}_z\cdot\hat{z})}_{-i\,\hbar} =$$

$$= i\,\hbar\,(\hat{y}\cdot\hat{p}_z - \hat{z}\cdot\hat{p}_y) = i\,\hbar\,\hat{L}_z$$

Solution II.

Use the formulas:

$$\hat{x} = x\cdot, \quad \hat{p}_n = \frac{\hbar}{i}\frac{\partial}{\partial x_n}$$

Then

$$\hat{L}_x = \left(y\frac{\hbar}{i}\frac{\partial}{\partial z} - z\frac{\hbar}{i}\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = \left(z\frac{\hbar}{i}\frac{\partial}{\partial x} - x\frac{\hbar}{i}\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = \left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$$

Substituting and performing the derivations where appropriate we will get the same result.

3. Consider a *free* electron (whose wave function is a plane wave) incident on the infinite planar surface whose equation is x = 0. In the region $x \geq 0$ there is a constant potential $V = 5eV$. The energy of the particle is $E = 5.6eV$ and the velocity of the particle is perpendicular to the surface of the plane. Determine the refraction index (i.e. the ratio of the velocities of the wave in the two regions). (**2 points**)

   Solution:

   The ratio of the velocities is the same as the ratio of momenta, which in turn is the same as the ratio of wave numbers. In both regions the total energy is higher than the potential therefore the wave function is a linear combination of terms

$$e^{i\,\hbar\,\mathbf{k}\,\mathbf{r}} = e^{i\mathbf{p}\,\mathbf{r}}$$

   where p and k are real numbers.
   Because the movement is perpendicular to the plane in question, which is is the $y - z$ plane the problem is one dimensional. The total energy

$$E_{tot} = \frac{p^2}{2m_e} + V(x)$$

so

$$p = \sqrt{2\, m_e \left(E_{tot} - V(x)\right)}$$

And $V(x) = 0$ *when* $x < 0$ *and* $V(x) = 5\, eV$ *when* $x \geq 0$.

$$n = \frac{p_1}{p_2} = \frac{\sqrt{E}}{\sqrt{E-V}} = \sqrt{\frac{E}{E-V}} = 3.056$$

4. The primitive vectors of the reciprocal lattice of a crystal are: $\mathbf{a}_1 = a \cdot (\mathbf{i} + \mathbf{j})$, $\mathbf{a}_2 = a \cdot (\mathbf{j} + \mathbf{k})$, and $\mathbf{a}_3 = a \cdot (\mathbf{k} + \mathbf{i})$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are Cartesian unit vectors. What type of lattice does the crystal have? (**2 points**)

   Solution:

   During the semester we saw that these primitive reciprocal vectors are the base primitive vectors of an fcc lattice so the corresponding direct lattice is a bcc lattice.

   Although no proof was required that the reciprocal lattice is fcc *we provide one here for your convenience*

   Proof :
   Any lattice point may be expressed as a linear combination of the primitive vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$*:*

   $\mathbf{r} = n_1\, \mathbf{a}_1 + n_2\, \mathbf{a}_2 + n_3\, \mathbf{a}_3 = a \cdot [n_1 \cdot (\mathbf{i} + \mathbf{j}) + n_2 \cdot (\mathbf{j} + \mathbf{k}) + n_3 \cdot (\mathbf{k} + \mathbf{i})]$

   where n_1, n_2, n_3 *are integers. Selecting the x-y plane as an example only the linear combination of* $\mathbf{i}$ *and* $\mathbf{k}$ *may have non 0 multipliers. From this:*

   $$\mathbf{r}_{on\, x\, y\, plane} = a\left[(n_1 + n_3) \cdot \mathbf{i} + (n_2 + n_3) \cdot \mathbf{k}\right] \qquad \textit{and}$$

   $$(n_1 + n_2) \cdot \mathbf{j} = 0 \qquad \Rightarrow \qquad n_1 = -n_2$$

   $$\mathbf{r}_{on\, x\, y\, plane} = a\left[(n_1 + n_3) \cdot \mathbf{i} + (n_3 - n_1) \cdot \mathbf{k}\right]\ , \textit{ where } \quad n_1, n_3 = 0,\ \pm 1,\ \pm 2, ...$$

   So the coordinates of the lattice points on the x-y plane may be calculated by solving the following Diophantine equations for n_1 *and* n_3*:*

   $$x = n_1 + n_3$$
   $$y = n_3 - n_1$$
   $$n_1 = \frac{x+y}{2}$$
   $$n_2 = \frac{x-y}{2}$$

   The points nearest to the origin in the first quadrant ($x \geq 0, y \geq 0$*) are*

   $$\begin{array}{ll} (0.0), & n_1 = 0, n_3 = 0 \\ (1,1), & n_1 = 0, n_3 = 1 \\ (2,0), & n_1 = 1, n_3 = 1 \\ (0,2), & n_1 = -1, n_3 = 1 \\ (2,2) & n_1 = 0, n_3 = 2 \end{array}$$

and these are the points on one face of a cubic fcc lattice. With similar derivation for the other faces we conclude that this is an fcc lattice.

5. A rod of P doped Si is 1 cm long and has a diameter of $1mm$. At room temperature, the intrinsic concentration in silicon is $n_i = 1.5 \cdot 10^{16}\, m^{-3}$. The electron and hole mobilities are $\mu_e = 0.13\, m^2\, V^{-1} s^{-1}$ and $\mu_h = 0.05\, m^2 V^{-1} s^{-1}$. The P concentration is $n_P = 1.5 \cdot 10^{17} m^{-3}$ Calculate the conductivity σ of the silicon and the resistance R of the rod. (**2 points**)

   Solution:

   P is a donor in Si, so a P doped Si is an n-type Si. Although this is a doped semiconductor the law of mass action still valid:

$$n_c \cdot p_v = n_c^i \cdot p_v^i = n_i^2$$

   The concentration of electrons in the conduction band and holes in the valence band are

$$n_c = N_d + n_i = 1.5 \cdot 10^{17} + 1.5\, 10^{16} = 1.65 \cdot 10^{17}\, m^{-3},$$

$$p_v = \frac{n_i^2}{n_c} = \frac{(1.5\, 10^{16})^2}{1.65 \cdot 10^{17}} = 1.36 \cdot 10^{15}\, m^{-3}$$

   The $j = |\mathbf{j}| = |\sigma\, \mathbf{E}|$ *current density is*

$$j = e(n_c \cdot \mu_e + p_v \cdot \mu_h)\, E = 1.6 \cdot 10^{-19} \cdot (1.65 \cdot 10^{17} \cdot 0.13 + 1.36 \cdot 10^{15} \cdot 0.05)\, E$$

$$j = 3.448 \cdot 10^{-3}\, E \qquad \Rightarrow \qquad \boxed{\sigma = 3.448 \cdot 10^{-3}\, \Omega^{-1}\, m^{-1}}$$

   The resistance of the rod is

$$R = \frac{1}{\sigma}\frac{l}{A} = \frac{10^{-4} \cdot 4}{3.448 \cdot 10^{-3} \cdot (10^{-3})^2\, \pi} = \boxed{2.5eM\Omega}$$